

## ON BOUNDS OF MATRIX EIGENVALUES

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**ABSTRACT.** In this paper, we give estimates for both upper and lower bounds of eigenvalues of a simple matrix. The estimates are shaper than the known results.

### 1. INTRODUCTION

As is well known, the eigenvalues of a matrix play an important role in solving linear systems [1, 3, 5], especially in the perturbation problems [2, 6]. The purpose of this note is to give a specific estimate of the eigenvalues.

Let  $A = (a_{ij})$  be an  $n \times n$  complex matrix with conjugate transpose  $A^*$ ,  $\bar{A}$  denote the conjugate, and  $\text{tr}A$  represent the trace of matrix  $A$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A$ , then

$$\sum_{i=1}^n |\lambda_i|^2 \leq \|A\|^2 = \sum_{i,j=1}^n |a_{ij}|^2 = \text{tr}(AA^*),$$

where  $\|A\|$  denotes the Frobenius norm of  $A$ . Let

$$\Re_A = \frac{A + A^*}{2},$$

$$\Im_A = \frac{A - A^*}{2i},$$

we call  $\Re_A$  the Hermitian real part and  $\Im_A$  the Hermitian imaginary part of  $A$ . Let

$$q_A = \|A\|^2 - \frac{|\text{tr}(A)|^2}{n},$$

$$\Delta_A = \frac{\|AA^* - A^*A\|^2}{2}.$$

### 2. MAIN THEOREM

**Theorem 2.1.** Suppose  $\lambda$  is an eigenvalue of an  $n \times n$  complex matrix  $A$  with geometric multiplicity  $t$ , then

$$\left| \lambda - \frac{\text{tr}(A)}{n} \right| \leq \sqrt{\frac{n-t}{(2n-t)t}} \sqrt{\frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2} \Delta_A}}. \quad (2.1)$$

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**Theorem 2.2.** Suppose  $\lambda_{\mathfrak{R}_A}, \lambda_{\mathfrak{S}_A}$  are the eigenvalues of an  $n \times n$  complex matrices  $\mathfrak{R}_A$  and  $\mathfrak{S}_A$  with geometric multiplicity  $t$ , respectively, then

$$\left| \lambda_{\mathfrak{R}_A} - \frac{\text{tr}(\mathfrak{R}_A)}{n} \right| \leq \sqrt{\frac{n-t}{nt} q_{\mathfrak{R}_A}}, \quad (2.2)$$

$$\left| \lambda_{\mathfrak{S}_A} - \frac{\text{tr}(\mathfrak{S}_A)}{n} \right| \leq \sqrt{\frac{n-t}{nt} q_{\mathfrak{S}_A}}. \quad (2.3)$$

### 3. PROOF OF THEOREM

Before giving the proof of Theorems 2.1 and 2.2, we present some lemmas.

**Lemma 3.1** (see [4]). Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of an  $n \times n$  complex matrix  $A$ , then

$$\sum_{j=1}^n |\lambda_j|^2 \leq \sqrt{\|A\|^4 - \Delta_A}.$$

**Lemma 3.2.** Let  $A$  be an  $n \times n$  complex matrix,  $\text{rank}(A)$  represent the rank of  $A$ . Then

$$|\text{tr}(A)|^2 \leq \text{rank}(A) \sqrt{\|A\|^4 - \Delta_A}.$$

*Proof.* Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A$ ,  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Suppose that the number of nonzero eigenvalues is  $k$ . Without loss of generality, we can denote the nonzero eigenvalues of  $A$  by  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Then it is easily seen that

$$k \leq \text{rank}(A).$$

Now suppose that  $R = \Lambda + M$  is a Schur triangular form of  $A$ , i.e.,  $A = U^* R U$ ,  $U$  is unitary orthogonal,  $\Lambda$  is diagonal and  $M$  is upper triangular. From Lemma 3.1, we have

$$\sum_{j=1}^k |\lambda_j|^2 \leq \sqrt{\|A\|^4 - \Delta_A}.$$

Then

$$|\text{tr}(A)|^2 = \left| \sum_{j=1}^k \lambda_j \right|^2 \leq k \sum_{j=1}^k |\lambda_j|^2 \leq \text{rank}(A) \sum_{j=1}^k |\lambda_j|^2 \leq \text{rank}(A) \sqrt{\|A\|^4 - \Delta_A}.$$

This shows the validity of conclusion.  $\square$

Next, we provide the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Let  $M = \lambda I - A$ , where  $I$  is the  $n \times n$  identity matrix,  $\lambda$  is the  $t$  multiple eigenvalue of  $A$ . Then we have

$$\text{rank}(M) = \text{rank}(\lambda I - A) \leq n - t,$$

and the following equality

$$\Delta_M = \frac{\|(\lambda I - A)(\bar{\lambda} I - A^*) - (\bar{\lambda} I - A^*)(\lambda I - A)\|}{2} = \Delta_A.$$

From Lemma 3.2, we have

$$\begin{aligned} |\operatorname{tr}(M)|^2 &\leq \operatorname{rank}(M) \sqrt{\|M\|^4 - \Delta_M} \\ &\leq (n-t) \sqrt{\|M\|^4 - \Delta_M} \leq (n-t) \sqrt{\|\lambda I - A\|^4 - \Delta_A}. \end{aligned} \quad (3.1)$$

In addition, by simple manipulations, we obtain

$$\begin{aligned} |\operatorname{tr}(\lambda I - A)|^2 &= \operatorname{tr}(\lambda I - A) \operatorname{tr}(\bar{\lambda} I - A^*) \\ &= n^2 |\lambda|^2 - n \lambda \operatorname{tr}(A^*) - n \bar{\lambda} \operatorname{tr}(A) + |\operatorname{tr}(A)|^2 = n\sigma + |\operatorname{tr}(A)|^2, \end{aligned} \quad (3.2)$$

where  $\sigma = n|\lambda|^2 - \lambda \operatorname{tr}(A^*) - \bar{\lambda} \operatorname{tr}(A)$ . Moreover,

$$\begin{aligned} \|\lambda I - A\|^4 &= (\operatorname{tr}((\lambda I - A)(\lambda I - A)^*))^2 \\ &= (n|\lambda|^2 - \lambda \operatorname{tr}(A^*) - \bar{\lambda} \operatorname{tr}(A) + \|A\|^2)^2 = (\sigma + \|A\|^2)^2. \end{aligned} \quad (3.3)$$

Eliminating  $\sigma$  from the formulae (3.2) and (3.3), we get

$$\|\lambda I - A\|^4 = \left( \frac{|\operatorname{tr}(\lambda I - A)|^2 - |\operatorname{tr}(A)|^2}{n} + \|A\|^2 \right)^2. \quad (3.4)$$

Let  $s = \left| \lambda - \frac{\operatorname{tr}(A)}{n} \right|^2$ ,  $q_A = \|A\|^2 - \frac{|\operatorname{tr}(A)|^2}{n}$ . Then

$$|\operatorname{tr}(\lambda I - A)|^2 = n^2 s, \quad (3.5)$$

and

$$\|\lambda I - A\|^4 = (ns + q_A)^2. \quad (3.6)$$

By substituting the equalities (3.5) and (3.6) into (3.1), it follows that

$$n^2 s \leq (n-t) \sqrt{(ns + q_A)^2 - \Delta_A}.$$

Consequently, by straightforward computations, we have

$$s = \left| \lambda - \frac{\operatorname{tr}(A)}{n} \right|^2 \leq \frac{n-t}{(2n-t)t} \left( \frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2} \Delta_A} \right).$$

The result follows immediately.  $\square$

*Proof of Theorem 2.2.* Notice that

$$\sqrt{\frac{n-t}{(2n-t)t}} \sqrt{\frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2} \Delta_A}} \leq \sqrt{\frac{n-t}{nt} q_A}.$$

Furthermore, the above equality holds if and only if  $\Delta_A = 0$ . In other words,  $A$  is normal, i.e.,  $AA^* = A^*A$ . By Theorem 2.1, and taking into account that  $\Re_A$  and  $\Im_A$  are both normal matrices, we get the validity of Theorem 2.2.  $\square$

*Remark.* In terms of estimates on bounds of the largest modulus eigenvalue  $|\lambda|_{\max}$  of matrix  $A$ , the following inequality was given in [7, 8],

$$\frac{|\operatorname{tr}(A)|}{n} \leq |\lambda|_{\max} \leq \frac{|\operatorname{tr}(A)|}{n} + \sqrt{\frac{n-1}{n} q_A}. \quad (3.7)$$

We note that the estimates (2.1), (2.2), (2.3) are sharper than (3.7) in some extent. That is to say, the results presented in this paper improve the ones given in [7, 8] partially, and can be taken as supplements to the conclusions known in [5, 7, 8], especially for the upper bound estimation of eigenvalues of a matrix.

## REFERENCES

- [1] O. AXELSSON, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1994.
- [2] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, 3rd Edition, The Johns Hopkins University Press, Baltimore and London, 1996.
- [3] A. GREENBAUM, *Iterative Methods for Solving Linear Systems*, SIAM, Philadelphia, PA, 1997.
- [4] R. KRESS, H. L. DE VRIES AND R. WEGMANN, *On nonnormal matrices*, Linear Algebra Appl., 8 (1974), pp. 109–120.
- [5] J. LIANG, *Distribution of matrix eigenvalue and its application in numerical analysis*, J. Uni. Petrol., 25 (2001), pp. 113–116.
- [6] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
- [7] H. WOLKOWICZ AND G. P. H. STYAN, *More bounds for eigenvalues using traces*, Linear Algebra Appl., 31 (1980), pp. 1–17.
- [8] H. WOLKOWICZ AND G. P. H. STYAN, *Bounds for eigenvalues using traces*, Linear Algebra Appl., 29 (1980), pp. 471–506.

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